IDENTITIES FOR HYPERELLIPTIC $\wp$-FUNCTIONS OF GENUS ONE, TWO AND THREE IN COVARIANT FORM.

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Abstract. We give a covariant treatment of the quadratic differential identities satisfied by the $\wp$-functions on the Jacobian of smooth hyperelliptic curves of genus ≤ 3.

1. Introduction

A classical problem in the theory of a planar (n, s) algebraic curve is a description of the differential equations satisfied by meromorphic, multiply periodic functions defined on its Jacobian variety. In the genus g hyperelliptic case ($n = 2$, $s = 2g+2$) the field of such functions is entirely described in terms of certain $\wp_{ij}$ functions which generalize the Weierstrass $\wp$-function on the elliptic curve, the genus one case.

The derivation of these identities has been a major concern over the last ten to fifteen years and many results have been published: see [8, 10, 11] for seminal literature.

The aim of this paper is to promote a new methodology which considerably simplifies the derivation and presentation of these identities by utilizing elementary representation theory. The fundamental observation is that the underlying algebraic curves belong to generic families permuted under an $\mathfrak{sl}_2$ action. This can be interpreted [2, 3] as a covariance property that translates into covariance of the $\wp$-function identities. This means that each polynomial identity between derivatives of the $\wp$-function belongs to a finite dimensional representation of $\mathfrak{sl}_2$, the knowledge of which depends only upon a highest weight element. It is only necessary to find these highest weight identities to generate the other identities in the representation.

However, a requirement of this approach is that we develop the theory for the generic member of the family of curves. This is in contrast to former treatments where a simpler, normal form is exploited by moving a branch point to infinity, i.e. removing the highest degree term.

The only case where the covariant equations are written down is for genus two hyperelliptic curves by Baker [5]. He achieves this by establishing the equations for the curve in normal form and then undoing the "normalizing" transformation's effect on the identities. Even so he finds it necessary to introduce a "fudge factor" to restore full covariance.

This "fudge factor" points to another problem. Not only must the curve be in general position but the fundamental (Kleinian) definition of the $\wp$-function [5, 6, 7, 8] must itself be rendered covariant. This problem reasserts itself in the next highest genus and the Baker equations for the genus three curve [6] are nowhere written down in covariant form.

The same issues occur in purely algebraic treatments, that of Cassels and Flynn for instance [9]. The formulation of their approach, important for curves over general fields, can also be rendered covariant and will be discussed in another publication. In this paper we work entirely over $\mathbb{C}$.

In this respect a note on the approach of the papers [2, 3] by the present author and collaborators is in order. What was attempted in those papers was a radically different approach to the analytic theory based on a very simple definition of the $\wp$-function, quite different to Klein's but with some philosophical proximity to that of [9]. However, whilst this was an effective approach to genus two, attempts so far to extend it to higher genus have foundered on finding the corresponding, simple definition of the $\wp$-function.

The programme of the current paper is, therefore, firstly to define the $\wp$ function in a covariant way and secondly to derive the identities it satisfies by combining the traditional technique of expansion about a chosen point with the Lie algebraic representation theory. We do this for genera one, two and three to recover known sets of differential equation or their equivalents. The emphasis is placed on the methodology.

The results so obtained are rather beautiful generalizations of the formulae found in [6, 7, 8]. Most of all we obtain a covariant bordered determinantal form of the set of quadratic identities in the $\wp_{ijk}$ for genus two, familiar from Baker's work [7], and a new generalization of this formula to the genus three case involving a doubly bordered determinant. These quadratic relations should presumably be regarded as the most fundamental differential identities and it is a positive feature of the covariant machinery that it produces them in a systematic manner at the simplest level.

2. Lie algebraic operations

Curves of the form

$$v(x, y; a_0, \ldots, a_{2g+2})) = y^2 - \sum_{i=0}^{2g+2} \binom{2g+2}{i} a_i x^i = 0 \tag{2.1}$$

are generically hyperelliptic and of genus g: that is, unless some special relations obtain between the coefficients.

The family of such curves is permuted under transformations given by

$$x \mapsto X = \frac{\alpha x + \beta}{(\gamma x + \delta)}, \tag{2.2}$$

$$y \mapsto Y = \frac{y}{(\gamma x + \delta)^{g+1}}, \tag{2.3}$$

where

$$\alpha\delta - \beta\gamma = 1,$$

mapping the above curve into

$$V(X, Y; A_0, \ldots, A_{2g+2}) = Y^2 - \sum_{i=0}^{2g+2} \binom{2g+2}{i} A_i X^i = 0 \tag{2.4}$$

the A_i being functions of the a_i and the parameters α, β, γ and δ.

This can be restated as infinitesimal *covariance* conditions,

$$\mathbf{e}v(x,y;a_0,\dots,a_{2g+2}) = 0 \tag{2.5}$$

$$\mathbf{f}v(x,y;a_0,\dots,a_{2g+2}) + 2(g+1)xv(x,y;a_0,\dots,a_{2g+2}) = 0 \tag{2.6}$$

where the generators $\mathbf{e}$ and $\mathbf{f}$ are given by

$$\mathbf{e} = \partial_x - \sum_{i=0}^{2g+2}(2g+2-i)a_{i+1}\partial_{a_i} \tag{2.7}$$

$$\mathbf{f} = -x^2\partial_x - (g+1)xy\partial_y - \sum_{i=0}^{2g+2} ia_{i-1}\partial_{a_i} \tag{2.8}$$

$$\mathbf{h} = -2x\partial_x - (2g+2)y\partial_y - \sum_{i=0}^{g+1} ia_i\partial_{a_i}. \tag{2.9}$$

These generators satisfy the $\mathfrak{sl}_2$ commutation relations,

$$[\mathbf{h},\mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h},\mathbf{f}] = -2\mathbf{f}, \quad [\mathbf{e},\mathbf{f}] = \mathbf{h}. \tag{2.10}$$

The coefficients $a_0, a_1, \dots, a_{2g+2}$ are a basis for a $2g+3$ dimensional representation.

The space of holomorphic differentials on the curve is spanned by the set

$$\{\frac{x^{i-1}dx}{y}|i = 1,\dots,g\}$$

and the symmetric sums of each of these differentials taken over g copies of the curve,

$$du_i = \sum_{j=1}^{g}\frac{x_j^{i-1}dx_j}{y_j} \tag{2.11}$$

are a basis for holomorphic one-forms on the Jacobian variety of the curve.

One checks the following action of $\mathfrak{sl}_2$:

$$\mathbf{e}du_i = (i-1)du_{i-1} \tag{2.12}$$

$$\mathbf{f}du_i = (g-i)du_{i+1} \tag{2.13}$$

and it then follows that

$$\mathbf{e}\partial_{u_i} = -i\partial_{u_{i+1}} \tag{2.14}$$

$$\mathbf{f}\partial_{u_i} = -(g-i+1)\partial_{u_{i-1}} \tag{2.15}$$

3. Covariant Klein relations

Our starting point will be the Kleinian definition of the doubly indexed $\wp$ functions: $\wp_{ij} = \wp_{ji}$ [8]. The indices are to be thought of as derivatives with respect to the variables u_i. There are thus integrability conditions of the form,

$$\wp_{ij,k} = \wp_{ik,j} = \wp_{kj,i} \quad \forall i,j,k. \tag{3.1}$$

For the moment we think of these objects purely as indexed symbols satisfying algebraic rules of differentiation and a set of identities to be specified shortly. However they are not traditionally defined in a covariant manner, that is in a way that

respects the further relations following from (2.14), namely,

$$\mathbf{e}\wp_{ij} = -i\wp_{i+1\,j} - j\wp_{i\,j+1} \tag{3.2}$$

$$\mathbf{f}\wp_{ij} = -(g-i+1)\wp_{i-1\,j} - (g-j+1)\wp_{i\,j-1} \tag{3.3}$$

In order to proceed we need to adjust the fundamental definition by adding correction terms without destroying the fundamental singularity properties of the $\wp_{ij}$.

How to do this is best seen by example and we explain it now for the genus two case.

The classical definition in genus two assumes a normal form with branch point at infinity, $a_6 = 0, a_5 = \frac{2}{3}$, and is:

$$\wp_{11} + (x_i + x)\wp_{12} + xx_i\wp_{22} = \frac{F(x, x_i) - yy_i}{4(x - x_i)^2} \tag{3.4}$$

where $i = 1, 2$ and $\wp$ is a function of the argument $\int^x d\mathbf{u} + \int^{x_1} d\mathbf{u} + \int^{x_2} d\mathbf{u}$, $\mathbf{u} = (u_1, u_2)$. The function $F(x, x_i)$ is the classical polar form

$$F(x, x_i) = 2(x+x_i)x^2x_i^2 + 15a_4x^2x_i^2 + 10a_3(x+x_i)xx_i + 15a_2xx_i + 3a_1(x+x_i) + a_0 \tag{3.5}$$

For the generic case one must clearly reinstate the coefficients a_6 and a_5 but this alone is not enough to render the equation covariant which, in this case means *invariant*, it being a single relation.

The left hand side becomes invariant on dividing by $x - x_i$ since both

$$(\wp_{11}, -2\wp_{12}, \wp_{22})$$

and

$$\mathbf{X^3} = \left(\frac{2xx_i}{x - x_i}, -\frac{x + x_i}{x - x_i}, \frac{2}{x - x_i}\right)$$

are three dimensional representations. Note that the x_i here can be either choice from x_1 and x_2.

On the right hand side the ratio $\frac{yy_i}{(x-x_i)^3}$ is now also seen to be invariant but $\frac{F(x,x_i)}{(x-x_i)^3}$ is not.

Note however that there is a seven dimensional representation,

$$\mathbf{X^7} = \left(\frac{6}{(x-x_i)^3}, -\frac{3(x+x_i)}{(x-x_i)^3}, \frac{3(x^2+3xx_i+x_i^2)}{(x-x_i)^3}, -\frac{(x^3+9x^2x_i+9x_i^2x+x^3)}{(x-x_i)^3}\right.,$$
$$\left.\frac{3(x^2+3xx_i+x_i^2)xx_i}{(x-x_i)^3}, -\frac{3(x+x_i)x^2x_i^2}{(x-x_i)^3}, \frac{6x^3x_i^3}{(x-x_i)^3}\right) \tag{3.6}$$

which, when taken with the coefficients $a_0, -a_1, a_2, -a_3, a_4, -a_5, a_6$ gives an invariant. This modification does not alter the fundamental requirement that in the limit $x \to x_i, y \to y_i$ the $\wp_{ij}$ are regular but have poles of order 2 when $x \to x_i, y \to -y_i$ [9]. Hence our modified definition is,

$$\wp_{11}\mathbf{X^3}_2 + \wp_{12}\mathbf{X^3}_1 + \wp_{22}\mathbf{X^3}_0 = \frac{\tilde{F}(x, x_i) - yy_i}{2(x - x_i)^3} \tag{3.7}$$

where

$$\frac{\tilde{F}(x, x_i)}{(x - x_i)^3} = a_0\mathbf{X^7}_6 + a_1\mathbf{X^7}_5 + a_2\mathbf{X^7}_4 + a_3\mathbf{X^7}_3 + a_4\mathbf{X^7}_2 + a_5\mathbf{X^7}_1 + a_6\mathbf{X^7}_0 \tag{3.8}$$

is a covariant "polar" form.

The corresponding generalizations for other genera are straightforward and depend on constructing $2g+3$ dimensional representations, $\mathbf{X}^{2g+3}$, by taking highest weight elements $(x-x_i)^{-(g+1)}$ for $\mathbf{e}$ and applying $\mathbf{f}$ successively, with appropriate normalizations.

Thus, for instance, for genus one we write,

$$\wp_{11} = \frac{\tilde{F}(x,x_i) - yy_i}{2(x-x_i)^2} \tag{3.9}$$

where, using $\mathbf{X^5}$,

$$\tilde{F}(x,x_i) = a_0 + 2a_1(x+x_i) + a_2(x^2 + xx_i + x_i^2) + a_3(x+x_i)xx_i + a_4x^2x_i^2. \tag{3.10}$$

The covariant polar form stands in a geometric relation to the hyperelliptic curve $y^2 - \sum_{i=0}^{2g+2}\binom{2g+2}{i}a_ix^i$ of degree $g+2$ not shared by the traditional polar form; namely, the curve $yy_i - \tilde{F}(x,x_i) = 0$ of degree $g+1$ is tangent to order $g+1$ to the hyperelliptic curve at the common point (x_i, y_i).

For the calculations which follow we put the defining relations into the convenient form

$$yy_i - \mathbf{x}^t h \mathbf{x}_i = 0 \tag{3.11}$$

where h is a $(g+2)\times(g+2)$ matrix whose entries depend only upon the a_i and the $\wp_{ij}$. The $\mathbf{x}$'s are $g+2$-vectors of monomials,e.g.

$$\mathbf{x}^t = (1, x, x^2, \ldots, x^g, x^{g+1}). \tag{3.12}$$

4. Differential relations in genus one

Here we give a new, covariant treatment of the most classical case of all: the Weierstrass $\wp$-function.

Covariance of the quartic curve

$$y^2 = a_0 + 4a_1x + 6a_2x^2 + 4a_3x^3 + a_4x^4 \tag{4.1}$$

under $\mathfrak{sl}_2(\mathbb{C})$ requires

$$\begin{aligned}
\mathbf{e}(x) &= 1\\
\mathbf{e}(y) &= 0\\
\mathbf{f}(x) &= -x^2\\
\mathbf{f}(y) &= -2xy\\
\mathbf{e}(a_i) &= -(4-i)a_{i+1}\\
\mathbf{f}(a_i) &= -ia_{i-1}
\end{aligned}$$

There is only one holomorphic differential on the curve: $du_1 = \frac{dx}{y}$ and it is clear that

$$\begin{aligned}
\mathbf{e}(du_1) &= 0\\
\mathbf{f}(du_1) &= 0
\end{aligned}$$

so that $\wp_{11}$, $\wp_{111}$, etc. are all invariant.

Even for this, the simplest case, it is necessary to make the Klein definition covariant before we start by using $\mathbf{X}^5$ as at the end of the last section. We apply the fundamental definition of Klein [8], written in the form

$$yy_1 - \mathbf{x}^t h \mathbf{x}_1 = 0 \tag{4.2}$$

where $\mathbf{x} = (1, x, x^2)$, $\mathbf{x_1} = (1, x_1, x_1^2)$ but where h is now the covariantly modified, three by three matrix

$$h = \begin{bmatrix} a_0 & 2a_1 & a_2 - 2\wp_{11} \\ 2a_1 & 4a_2 + 4\wp_{11} & 2a_3 \\ a_2 - 2\wp_{11} & 2a_3 & a_4 \end{bmatrix} \tag{4.3}$$

Note that in terms of entries of h,

$$\begin{aligned} y^2 &= a(x) \\ &= h_{33}x^4 + (h_{32} + h_{23})x^3 + (h_{31} + h_{22} + h_{13})x^2 \\ &\quad + (h_{12} + h_{21})x + h_{11} \end{aligned}$$

each coefficient being independent of the $\wp_{11}$ symbol.

Take the residue of (4.2) at $x = \infty$, $y = \sqrt{h}_{33}(x^2 + \frac{h_{32}}{h_{33}}x + \ldots)$:

$$\sqrt{h}_{33}y_1 - h_{31} - h_{32}x_1 - h_{33}x_1^2 = 0 \tag{4.4}$$

The two index symbol $\wp_{11}$ is [8] a function of x and x_1 in the form

$$\wp_{11} = \wp_{11}\left(\int^x d\mathbf{u} + \int^{x_1} d\mathbf{u}\right) \tag{4.5}$$

Hence the effect of the operator $y\partial_x = \partial_{u_1}$ etc. on $\wp_{11}$ is

$$y\partial_x\wp_{11} = \wp_{111} \tag{4.6}$$

$$y_1\partial_{x_1}\wp_{11} = \wp_{111} \tag{4.7}$$

Now apply $y\partial_x$ to the Klein relation (4.2):

$$yy'y_1 - y\mathbf{x}'^t h\mathbf{x}_1 = \mathbf{x}^t(\partial_{u_1}h)\mathbf{x}_1 \tag{4.8}$$

Use of the defining relation allows us to replace yy_1 to give:

$$(y'\mathbf{x}^t - y{\mathbf{x}'}^t)h\mathbf{x}_1 = \mathbf{x}^t(\partial_{u_1}h)\mathbf{x}_1 \tag{4.9}$$

The highest order term using $y = \sqrt{h}_{33}(x^2 + \frac{h_{32}}{h_{33}}x + \ldots)$, yields

$$h_{33}(h\mathbf{x}_1)_2 - h_{23}(h\mathbf{x}_1)_3 = \sqrt{h_{33}}(\partial_{u_1}h\mathbf{x}_1)_3 \tag{4.10}$$

where we have used subscripts $(\cdot)_2$ and $(\cdot)_3$ to denote the second and third components of a vector quantity.

Explicitly we have the identity:

$$\begin{vmatrix} h_{12} & h_{13} \\ h_{23} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} x_1 + 2\sqrt{h}_{33}\wp_{111} = 0$$

The same identity arises if we differentiate the Klein relation with respect to x_1.

So far then y_1 is given by a quadratic in x_1, linear in $\wp_{11}$, and $\wp_{111}$ by a relation linear in x_1 and $\wp_{11}$. One further relation is afforded by the fact that (x_1, y_1) lies on the curve. Using the expression (4.4) for y_1 this becomes

$$\begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} x_1^2 + 2\begin{vmatrix} h_{12} & h_{13} \\ h_{32} & h_{33} \end{vmatrix} x_1 + \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} = 0. \tag{4.11}$$

We now eliminate x_1 between this quadratic relation and the preceeding linear expression for $\wp_{111}$. We obtain

$$\wp_{111}^2 = -\frac{1}{4}\begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \tag{4.12}$$

Identifying as customary the classical $\wp$-function with $\wp_{11}$ and its derivative, $\wp'$, with $\wp_{111}$ we have, expanding the determinant, the equation for the $\wp$-function for the *generic* curve of genus one:

$$\begin{aligned} \wp'^2 - 4\wp^3 \quad = \quad & -(a_0a_4 - 4a_1a_3 + 3a_2^2)\wp \\ & -a_0a_2a_4 + a_0a_3^2 - 2a_1a_2a_3 + a_2^3 + a_1^2a_4 \end{aligned} \tag{4.13}$$

4.1. **Remarks.**

4.1.1. The coefficients $a_0a_4 - 4a_1a_3 + 3a_2^2$ and $-a_0a_2a_4 + a_0a_3^2 - 2a_1a_2a_3 + a_2^3 + a_1^2a_4$ are readily verified to be invariants under the $\mathfrak{sl}_2(\mathbb{C})$ action. This is only to be expected from the classical approach. They sit inside the two-fold and three-fold tensor products of the five dimensional representation spanned by $\{a_0, a_1, a_2, a_3, a_4\}$.

4.1.2. Specializing to the case where one branch point is moved to ∞, we take $a_4 = 0$. By shifting x we can set $a_2 = 0$ and by scaling, set $a_3 = 1$:

$$\wp'^2 = 4\wp^3 + 4a_1\wp + a_0$$

Traditionally one associates this curve with the cubic

$$y^2 = 4x^3 + 4a_1x + a_0$$

parametrized by setting $x = \wp$ and $y = \wp'$ but we see that in fact the origin of the factor of 4 on the left hand side is not at all related to the value of a_3. It is rather an intrinsic value that holds for the generic curve. We could of course solve the relations obtained in the previous section to obtain x_1 and y_1 as functions of $\wp_{11}$, $\wp_{111}$ and the a_i inorder to parametrise the generic quartic, $y^2 = a(x)$. This parametrization looks, at first sight, rather unattractive although it reduces to the classical one when the branch point is moved to ∞.

4.1.3. The generic differential equation for the $\wp$-function above is actually what for higher genus would be called a quadratic identity. Consequently the coefficients in the differential equation are polynomial in the a_i and not linear.

4.1.4. Why is life more complicated for higher genus? Simply because the $\wp_{ij}$ are now a $\frac{1}{2}g(g+1)$ dimensional (not, in general, irreducible) representation and so their relations cannot be constructed solely from invariant quantities.

5. Differential relations in genus two

The fundamental definition of Klein can be modified to the form

$$yy_i - \mathbf{x}h\mathbf{x}_i^T = 0 \tag{5.1}$$

for $i = 1, 2$, where $\mathbf{x} = (1, x, x^2, x^3)$, $\mathbf{x_i} = (1, x_i, x_i^2, x_i^3)$ and h is the covariant four by four matrix

$$h = \begin{bmatrix} a_0 & 3a_1 & 3a_2 - 2\wp_{11} & a_3 - 2\wp_{12} \\ 3a_1 & 9a_2 + 4\wp_{11} & 9a_3 + 2\wp_{12} & 3a_4 - 2\wp_{22} \\ 3a_2 - 2\wp_{11} & 9a_3 + 2\wp_{12} & 9a_4 + 4\wp_{22} & 3a_5 \\ a_3 - 2\wp_{12} & 3a_4 - 2\wp_{22} & 3a_5 & a_6 \end{bmatrix} \tag{5.2}$$

Note that in terms of entries of h,

$$\begin{aligned} y^2 &= a(x) \\ &= h_{44}x^6 + (h_{34} + h_{43})x^5 + (h_{24} + h_{33} + h_{42})x^4 \\ &\quad +(h_{14} + h_{23} + h_{32} + h_{41})x^3 \\ &\quad +(h_{13} + h_{22} + h_{31})x^2 + (h_{12} + h_{21})x + h_{11} \end{aligned}$$

each coefficient being independent of the $\wp_{ij}$ symbols.

Take the residue of (5.1) at $x = \infty$, $y = \sqrt{h}_{44}(x^3 + \frac{h_{34}}{h_{44}}x^2 + \ldots)$:

$$\sqrt{h}_{44}y_1 - h_{41} - h_{42}x_1 - h_{43}x_1^2 - h_{44}x_1^3 = 0 \tag{5.3}$$

The two index symbols, $\wp_{ij}$ are [8] functions of x, x_1 and x_2 in the form

$$\wp_{ij} = \wp_{ij}\left(\int^x d\mathbf{u} + \int^{x_1} d\mathbf{u} + \int^{x_2} d\mathbf{u}\right) \tag{5.4}$$

Hence the effect of the operators $y\partial_x = \partial_{u_1} + x\partial_{u_2}$ etc. on the $\wp_{ij}$ is

$$y\partial_x\wp_{ij} = \wp_{ij1} + x\wp_{ij2} \tag{5.5}$$

$$y_1\partial_{x_1}\wp_{ij} = \wp_{ij1} + x_1\wp_{ij2} \tag{5.6}$$

$$y_2\partial_{x_2}\wp_{ij} = \wp_{ij1} + x_2\wp_{ij2} \tag{5.7}$$

Apply $y_2\partial_{x_2}$ to the Klein relation (5.1) with $i = 1$. By elementary algebra it reduces, for all x, to the form

$$-2(x - x_1)^2\,(A + xB) = 0 \tag{5.8}$$

where A and B are functions of x_1, x_2 and the $\wp_{ijk}$. As there can be no relation linear in x between these objects [5], both the coefficients A and B must vanish:

$$\begin{aligned} \wp_{111} + (x_1 + x_2)\wp_{112} + x_1x_2\wp_{122} &= 0 \\ \wp_{112} + (x_1 + x_2)\wp_{122} + x_1x_2\wp_{222} &= 0 \end{aligned} \tag{5.9}$$

Now apply $y\partial_x$ to the Klein relation (5.1) with $i = 1$:

$$yy'y_1 - y\mathbf{x}'h\mathbf{x}_1^T = \mathbf{x}(\partial_{u_1}h + x\partial_{u_2})\mathbf{x}_1^T \tag{5.10}$$

Use of the defining relation allows us to replace yy_1 to give:

$$(y'\mathbf{x}^T - y\mathbf{x}'^T)h\mathbf{x}_1^T = \mathbf{x}(\partial_{u_1}h + x\partial_{u_2})\mathbf{x}_1^T \tag{5.11}$$

Using $y = \sqrt{h}_{44}(x^3 + \frac{h_{34}}{h_{44}}x^2 + \ldots)$, the highest order term yields

$$h_{44}(h\mathbf{x}_1^T)_3 - h_{34}(h\mathbf{x}_1^T)_4 = \sqrt{h_{44}}(\partial_{u_2}h\mathbf{x}_1^T)_4 \tag{5.12}$$

where again we have used subscripts $(\cdot)_i$ to denote ith components of a vector quantity.

Explicitly we have a quadratic identity:

$$\begin{vmatrix} h_{31} & h_{34} \\ h_{41} & h_{44} \end{vmatrix} + \begin{vmatrix} h_{32} & h_{34} \\ h_{42} & h_{44} \end{vmatrix} x_1 + \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} x_1^2 \\ +2\sqrt{h}_{44}(\wp_{122} + x_1\wp_{222}) = 0 \tag{5.13}$$

By the general symmetry of the problem the same identity must be satisfied by x_2. Thus we can obtain expressions for the symmetric combinations $x_1 + x_2$ and x_1x_2, namely:

$$2\sqrt{h}_{44}\wp_{222} = -\begin{vmatrix} h_{32} & h_{34} \\ h_{42} & h_{44} \end{vmatrix} - \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix}(x_1 + x_2) \tag{5.14}$$

$$2\sqrt{h}_{44}\wp_{122} = -\begin{vmatrix} h_{31} & h_{34} \\ h_{41} & h_{44} \end{vmatrix} + \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} x_1x_2 \tag{5.15}$$

Eliminating these symmetric combinations from the second of the pair (5.9) we obtain the relation:

$$\begin{vmatrix} h_{33}\wp_{112} - h_{32}\wp_{122} + h_{31}\wp_{222} & h_{34} \\ & \\ h_{43}\wp_{112} - h_{42}\wp_{122} + h_{41}\wp_{222} & h_{44} \end{vmatrix} = 0$$

from which it follows that

$$h_{33}\wp_{112} - h_{32}\wp_{122} + h_{31}\wp_{222} = \lambda h_{34} \tag{5.16}$$

$$h_{43}\wp_{112} - h_{42}\wp_{122} + h_{41}\wp_{222} = \lambda h_{44} \tag{5.17}$$

λ being some constant to be determined.

All the elements of these identities belong to irreducible representations of $\mathfrak{sl}_2$ and it is easy to show that the identities are mutually self-consistent under the Lie algebra action if λ is identified with $\wp_{111}$. They then form two of a multiplet of four identities (a four dimensional representation of $\mathfrak{sl}_2$) summarized in matrix form as,

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \begin{pmatrix} \wp_{222} \\ -\wp_{122} \\ \wp_{112} \\ -\wp_{111} \end{pmatrix} = 0 \tag{5.18}$$

An immediate consequence of this is the relation for the Kummer surface, quartic in the $\wp_{ij}$:

$$\det(h) = 0 \tag{5.19}$$

But, more than this, it follows (we do not give the argument here because it is a simplification of that leading up to equation (6.24) for the genus three case) straightforwardly from (5.18) and the theory of diagonalisation of the symmetric matrix h that if we define the bordered matrix,

$$H = \begin{pmatrix} h_{11} & -h_{12} & h_{13} & -h_{14} & l_0 \\ -h_{21} & h_{22} & -h_{23} & h_{24} & l_1 \\ h_{31} & -h_{32} & h_{33} & -h_{34} & l_2 \\ -h_{41} & h_{42} & -h_{43} & h_{44} & l_3 \\ l_0 & l_1 & l_2 & l_3 & 0 \end{pmatrix} \tag{5.20}$$

then $\det(H)$ is, up to a factor, the expression $(l_0\wp_{222} + l_1\wp_{122} + l_2\wp_{112} + l_3\wp_{111})^2$.

That this factor is, in fact, $-\frac{1}{4}$ could be established by the classical argument of singularity balancing between the leading terms, quadratic in the $\wp_{ijk}$ and cubic in the $\wp_{ij}$. However it is instructive and in keeping with the current, purely algebraic, philosophy to establish the result by using the relation arising by application of $y_1\partial_{x_1}$ to the Klein relation (5.1) for $i = 1$.

Immediately we have

$$yy_1y_1' - \mathbf{x}(\partial_{u_1}h + x_1\partial_{u_2}h)\mathbf{x_1}^T - y_1\mathbf{x}h\mathbf{x_1'}^T = 0 \tag{5.21}$$

Replacing y_1y_1' by $\frac{1}{2}a'(x_1)$, taking the $x = \infty$ residue of

$$\frac{1}{2}ya'(x_1) - y_1\mathbf{x}h\mathbf{x_1'}^T = \mathbf{x}(\partial_{u_1}h + x_1\partial_{u_2}h)\mathbf{x_1}^T \tag{5.22}$$

and by elimination of y_1 as before, we find:

$$\frac{1}{2}\left((h\mathbf{x}_1^T)_4^2 - h_{44}a(x_1)\right)' = 2\sqrt{h}_{44}(\wp_{112} + 2\wp_{122}x_1 + \wp_{222}x_1^2)$$

prime denoting differentiation with respect to x_1: that is, we ignore the implicit x_1 dependence of the $\wp_{ij}$. In fact the right hand side of this equation is easily seen to be cubic in x_1 and not, as at first sight it appears, quintic. Exploiting the symmetry of h gives us,

$$\begin{aligned} \sqrt{h}_{44}(\wp_{112} + 2\wp_{122}x_1 + \wp_{222}x_1^2) \quad + \quad & \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} x_1^3 + \frac{3}{2}\begin{vmatrix} h_{23} & h_{24} \\ h_{43} & h_{44} \end{vmatrix} x_1^2 \\ & + \left(\begin{vmatrix} h_{13} & h_{14} \\ h_{43} & h_{44} \end{vmatrix} + \frac{1}{2}\begin{vmatrix} h_{22} & h_{24} \\ h_{42} & h_{44} \end{vmatrix}\right) x_1 \\ & + \frac{1}{2}\begin{vmatrix} h_{12} & h_{14} \\ h_{42} & h_{44} \end{vmatrix} = 0 \end{aligned} \tag{5.23}$$

It is straightforward to eliminate (5.13) from the above to leave a second quadratic identity,

$$2\sqrt{h}_{44}(\wp_{112} - \wp_{222}x_1^2) + \begin{vmatrix} h_{23} & h_{24} \\ h_{43} & h_{44} \end{vmatrix} x_1^2 + \begin{vmatrix} h_{22} & h_{24} \\ h_{42} & h_{44} \end{vmatrix} x_1 + \begin{vmatrix} h_{12} & h_{14} \\ h_{42} & h_{44} \end{vmatrix} = 0$$

Again, eliminating x_1^2 between this and (5.13) provides a relation of degree one in x_1. Since the x_i can satisfy nothing simpler than quadratic relations the coefficient of x_1 and the constant term must be identically zero. The first is

$$4h_{44}\wp_{222}^2 = \begin{vmatrix} h_{32} & h_{34} \\ h_{42} & h_{44} \end{vmatrix}\begin{vmatrix} h_{23} & h_{24} \\ h_{43} & h_{44} \end{vmatrix} - \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix}\begin{vmatrix} h_{22} & h_{24} \\ h_{42} & h_{44} \end{vmatrix} \tag{5.24}$$

and by a well-known identity for 3×3 determinants [1]:

$$-4\wp_{222}^2 = \begin{vmatrix} h_{22} & h_{23} & h_{24} \\ h_{32} & h_{33} & h_{34} \\ h_{42} & h_{43} & h_{44} \end{vmatrix} \tag{5.25}$$

This allows us to fix the value of the constant of proportionality and we obtain a beautiful, covariant generalization of Baker's formula [7]:

$$(l_0\wp_{222}+l_1\wp_{122}+l_2\wp_{112}+l_3\wp_{111})^2=-\frac{1}{4}\begin{vmatrix} h_{11} & -h_{12} & h_{13} & -h_{14} & l_0\\ -h_{21} & h_{22} & -h_{23} & h_{24} & l_1\\ h_{31} & -h_{32} & h_{33} & -h_{34} & l_2\\ -h_{41} & h_{42} & -h_{43} & h_{44} & l_3\\ l_0 & l_1 & l_2 & l_3 & 0 \end{vmatrix}$$

For later comparison we change the sign of l_1 and l_3:

$$(l_0\wp_{222}-l_1\wp_{122}+l_2\wp_{112}-l_3\wp_{111})^2=-\frac{1}{4}\begin{vmatrix} h_{11} & h_{12} & h_{13} & h_{14} & l_0\\ h_{21} & h_{22} & h_{23} & h_{24} & l_1\\ h_{31} & h_{32} & h_{33} & h_{34} & l_2\\ h_{41} & h_{42} & h_{43} & h_{44} & l_3\\ l_0 & l_1 & l_2 & l_3 & 0 \end{vmatrix}$$

In section 7 we will derive identities linear in the $\wp_{ij}$ and $\wp_{ijk}$ from the above quadratic identities. *Presumably* all $\wp$-function identities arise from these quadratic ones by algebraic and differential processes but, of course, this is not immediately clear. Nor is it immediately essential to their application in this paper.

6. Differential relations in genus three.

The last section recovers classical results in that the covariant identities were written down in [7] though not there derived in a covariant manner. By contrast a covariant treatment of higher genus hyperelliptic (or non-hyperelliptic) curves has not been given before. This we now do.

For genus three we have three covariant Klein equations:

$$yy_i-\mathbf{x}h\mathbf{x}_i^T=0 \tag{6.1}$$

for $i=1,2,3$, where $\mathbf{x}=(1,x,x^2,x^3,x^4)$, $\mathbf{x_i}=(1,x_i,x_i^2,x_i^3,x_i^4)$ and h is the 5×5 matrix,

$$\begin{bmatrix} a_0 & 4a_1 & 6a_2-2\wp_{11} & 4a_3-2\wp_{12} & a_4-2\wp_{13}\\ 4a_1 & 16a_2+4\wp_{11} & 24a_3+2\wp_{12} & 16a_4-2\wp_{22}+4\wp_{13} & 4a_5-2\wp_{23}\\ 6a_2-2\wp_{11} & 24a_3+2\wp_{12} & 36a_4+4\wp_{22}-4\wp_{13} & 24a_5+2\wp_{23} & 6a_6-2\wp_{33}\\ 4a_3-2\wp_{12} & 16a_4-2\wp_{22}+4\wp_{13} & 24a_5+2\wp_{23} & 16a_6+4\wp_{33} & 4a_7\\ a_4-2\wp_{13} & 4a_5-2\wp_{23} & 6a_6-2\wp_{33} & 4a_7 & a_8 \end{bmatrix}$$

The residue of (6.1) at $x=\infty$, $y(x)=\sqrt{h}_{55}x^4+\frac{h_{45}+h_{54}}{2\sqrt{h}_{55}}x^3\ldots$ gives

$$\sqrt{h}_{5,5}y_i-(h\mathbf{x}_i^T)_5=0, \tag{6.2}$$

for i with value 1, 2 or 3.

This time the operator $y\partial_x$ and its indexed relatives is given by $\partial_{u_1}+x\partial_{u_2}+x^2\partial_{u_3}$ etc. We may apply $y_2\partial_{x_2}$ to (6.1) with $i=1$ and take the residue at $x=\infty$ to obtain,

$$\left((\frac{\partial h}{\partial u_1}+x_2\frac{\partial h}{\partial u_2}+x_2^2\frac{\partial h}{\partial u_3})\mathbf{x}_1^T\right)_5=0 \tag{6.3}$$

Simplifying, removing overall factors of x_1-x_2, gives

$$\begin{aligned}\wp_{113}+(x_1+x_2)\wp_{123}+x_1x_2\wp_{223}+(x_1^2+x_2^2)\wp_{133}&\\ +(x_1+x_2)x_1x_2\wp_{233}+x_1^2x_2^2\wp_{333}&=0\end{aligned} \tag{6.4}$$

and, by cyclic interchange of the x_i,

$$\begin{aligned} \wp_{113}+(x_2+x_3)\wp_{123}+x_2x_3\wp_{223}+(x_2^2+x_3^2)\wp_{133} & \\ +(x_2+x_3)x_2x_3\wp_{233}+x_2^2x_3^2\wp_{333} &= 0 \end{aligned} \tag{6.5}$$

$$\begin{aligned} \wp_{113}+(x_3+x_1)\wp_{123}+x_3x_1\wp_{223}+(x_3^2+x_1^2)\wp_{133} & \\ +(x_3+x_1)x_3x_1\wp_{233}+x_3^2x_1^2\wp_{333} &= 0. \end{aligned} \tag{6.6}$$

From these three identities we can form three identities whose coefficients are symmetric functions in the x_i, namely:

$$\wp_{223}-\wp_{133}+s^{(1)}\wp_{233}+s^{(2)}\wp_{333} = 0 \tag{6.7}$$

$$\wp_{123}+s^{(1)}\wp_{133}-s^{(3)}\wp_{333}=0 \tag{6.8}$$

$$\wp_{113}-s^{(2)}\wp_{133}-s^{(3)}\wp_{233}=0 \tag{6.9}$$

where $s^{(1)}=x_1+x_2+x_3$, $s^{(2)}=x_1x_2+x_2x_3+x_3x_1$ and $s^{(3)}=x_1x_2x_3$.

An important observation at this point is that these three equations are overdetermined for $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$ so that the $\wp_{ijk}$ must satisfy the quadratic identity

$$\wp_{113}\wp_{333}-\wp_{123}\wp_{233}+\wp_{223}\wp_{133}-\wp_{133}^2=0. \tag{6.10}$$

This relation is in the kernel of $\mathbf{e}$ and thus is a highest weight element for a set of relations forming a five dimensional representation:

$$\begin{aligned} P_{\mathbf{5}}(0) &= \wp_{113}\wp_{333}-\wp_{123}\wp_{233}+\wp_{223}\wp_{133}-\wp_{133}^2 \\ P_{\mathbf{5}}(1) &= -\wp_{233}\wp_{113}-\wp_{112}\wp_{333}-\wp_{133}\wp_{222}+2\wp_{133}\wp_{123}+\wp_{233}\wp_{122} \\ P_{\mathbf{5}}(2) &= \wp_{133}\wp_{122}-\wp_{133}\wp_{113}-\wp_{223}\wp_{122}+\wp_{223}\wp_{113} \\ &\quad +\wp_{111}\wp_{333}+\wp_{123}\wp_{222}-2\wp_{123}^2 \\ P_{\mathbf{5}}(3) &= -\wp_{233}\wp_{111}-\wp_{112}\wp_{133}+\wp_{112}\wp_{223}-\wp_{113}\wp_{222}+2\wp_{113}\wp_{123} \\ P_{\mathbf{5}}(4) &= -\wp_{123}\wp_{112}+\wp_{113}\wp_{122}-\wp_{113}^2+\wp_{133}\wp_{111} \end{aligned}$$

This gives a set of five identities quadratic in the $\wp_{ijk}$, $P_{\mathbf{5}}(i)=0$ for $i=0,\dots 4$.

Differentiating (6.1) with respect to $y\partial_x$,

$$(y'(x)\mathbf{x}-y(x)\mathbf{x}')h\mathbf{x}_1^T=\mathbf{x}(\frac{\partial h}{\partial u_1}+x\frac{\partial h}{\partial u_2}+x^2\frac{\partial h}{\partial u_3})\mathbf{x}_1^T \tag{6.11}$$

we again use the expansion near $x=\infty$,

$$y(x)=\sqrt{h}_{55}x^4+\frac{h_{45}+h_{54}}{2\sqrt{h}_{55}}x^3\dots$$

collecting the highest order term (degree 6) in the identity just obtained. Thus, using the symmetry of the matrix h,

$$\begin{vmatrix} (h\mathbf{x}_1^T)_4 & (h\mathbf{x}_1^T)_5 \\ h_{54} & h_{55} \end{vmatrix} = (\frac{\partial h}{\partial u_3}\mathbf{x}_1^T)_5. \tag{6.12}$$

This is an identity cubic in x_1 also satisfied by x_2 and x_3:

$$\begin{aligned} \begin{vmatrix} h_{44} & h_{45} \\ h_{54} & h_{55} \end{vmatrix} x_i^3+\begin{vmatrix} h_{34} & h_{35} \\ h_{54} & h_{55} \end{vmatrix} x_i^2+\begin{vmatrix} h_{24} & h_{25} \\ h_{44} & h_{55} \end{vmatrix} x_i+\begin{vmatrix} h_{14} & h_{15} \\ h_{54} & h_{55} \end{vmatrix} & \\ +2\sqrt{h}_{55}(\wp_{133}+x_i\wp_{233}+x_i^2\wp_{333}) &= 0 \end{aligned} \tag{6.13}$$

We can now eliminate the symmetric functions $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$ in (6.7). From the first relation we obtain,

$$\begin{aligned} h_{24}\wp_{333} - h_{34}\wp_{233} + h_{44}(\wp_{223} - \wp_{133}) - h_{54}\lambda &= 0 \\ h_{25}\wp_{333} - h_{35}\wp_{233} + h_{45}(\wp_{223} - \wp_{133}) - h_{55}\lambda &= 0 \end{aligned} \tag{6.14}$$

where λ is an undetermined multiplier.

Since these identities are polynomial in the $\wp_{ijk}$ and the h_{ij} only they must belong to a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Application of the $\mathbf{e}$ and $\mathbf{f}$ operators must generate further identities. This can only work for a special value of λ and application of $\mathbf{e}$ to the second of the above identities shows that it will be a highest weight element (in the kernel of $\mathbf{e}$) only if $\lambda = \wp_{222} - 2\wp_{123}$. Hence we have

$$\begin{aligned} h_{24}\wp_{333} - h_{34}\wp_{233} + h_{44}(\wp_{223} - \wp_{133}) - h_{54}(\wp_{222} - 2\wp_{123}) &= 0 \\ h_{25}\wp_{333} - h_{35}\wp_{233} + h_{45}(\wp_{223} - \wp_{133}) - h_{55}(\wp_{222} - 2\wp_{123}) &= 0 \end{aligned} \tag{6.15}$$

We label the second of these identities $P_{\mathbf{9}}(0)$ because it is highest weight for a nine dimensional representation generated by repeated application of $\mathbf{f}$, a set of nine linearly independent identities $P_{\mathbf{9}}(i)$ for $i = 0, \ldots 8$. The last of these identities is:

$$P_{\mathbf{9}}(8) = h_{11}(\wp_{222} - 2\wp_{123}) - h_{12}(\wp_{122} - \wp_{113}) + h_{13}\wp_{112} - h_{14}\wp_{111} = 0 \tag{6.16}$$

Rather than write these out in detail now we shall summarize them in a more compact form shortly.

Now from a linear combination of the first of the identities (6.15) and $P_{\mathbf{9}}(1)$ we can form the highest weight identity for a seven dimensional representation:

$$\begin{aligned} P_{\mathbf{7}}(0) = -4h_{15}\wp_{333} + 4h_{35}\wp_{133} - h_{45}(2\wp_{123} + \wp_{222}) + 4h_{55}(\wp_{122} - \wp_{113}) & \\ -h_{34}\wp_{233} + h_{24}\wp_{333} - h_{44}(\wp_{133} - \wp_{223}) &= 0 \end{aligned} \tag{6.17}$$

Proceeding in this way with the other identities obtained from eliminating the symmetric functions from the other identities in (6.7), we obtain representations $P_{\mathbf{5}}$, $P_{\mathbf{3}}$ and $P_{\mathbf{1}}$, giving a total of $9 + 7 + 5 + 3 + 1 = 5^2$ relations linear in the three index symbols.

These identities are not presented in the simplest form however. They can be rendered more transparent by taking various linear combinations so that one only ever has four h terms arising in each identity. We do not give the details here because it involves routine linear algebra applied to the above identities, best accomplished using a computer algebra package. The *fact* that this simplification is possible, however, is of significance and it not clear to the present author exactly why it should be so.

After this rearrangement the identities take the form of a matrix product

$$hA = 0 \tag{6.18}$$

of the symmetric 5×5 matrix h and an antisymmetric 5×5 matrix

(6.19)
$$A=\left[\begin{array}{ccccc} 0 & -\wp_{333} & \wp_{233} & -\wp_{223}+\wp_{133} & \wp_{222}-2\wp_{123} \\ \wp_{333} & 0 & -\wp_{133} & \wp_{123} & -\wp_{122}+\wp_{113} \\ -\wp_{233} & \wp_{133} & 0 & -\wp_{113} & \wp_{112} \\ \wp_{223}-\wp_{133} & -\wp_{123} & \wp_{113} & 0 & -\wp_{111} \\ -\wp_{222}+2\wp_{123} & \wp_{122}-\wp_{113} & -\wp_{112} & \wp_{111} & 0 \end{array}\right]$$

One checks that the matrix A has rank at most three by virtue of the relations (6.11) obtained earlier. In fact the 4×4 minors of A are products of the $P_{\mathbf{5}}(i)$:

$$A(i,j)=P_{\mathbf{5}}(5-i)P_{\mathbf{5}}(5-j) \tag{6.20}$$

Further the 3×3 minors also have the $P_{\mathbf{5}}(i)$ as factors. There are however non vanishing 2×2 minors so the rank of A is exactly two.

Consequently the five by five matrix h has exactly a two dimensional zero eigenspace and, being symmetric, must be similar to a diagonal matrix of form $h_D=Diag(0,0,h_3,h_4,h_5)$.

We can use this fact to obtain identities quadratic in the $\wp_{ijk}$ by generalizing the argument in the genus two case as follows.

Let Π be the matrix which diagonalizes h, let $\mathbf{l}$ and $\mathbf{k}$ be arbitrary five component column vectors, I_2 the two by two identity matrix and consider

$$\begin{aligned}\left|\left[\begin{array}{cc}\Pi^T & 0\\ 0 & I_2\end{array}\right]\left[\begin{array}{ccc} h & \mathbf{l} & \mathbf{k}\\ \mathbf{l^T} & 0 & 0\\ \mathbf{k^T} & 0 & 0\end{array}\right]\left[\begin{array}{cc}\Pi & 0\\ 0 & I_2\end{array}\right]\right| &= \left|\left[\begin{array}{ccc} h_D & \Pi^T\mathbf{l} & \Pi^T\mathbf{k}\\ \mathbf{l^T}\Pi & 0 & 0\\ \mathbf{k^T}\Pi & 0 & 0\end{array}\right]\right| \\ &= h_3h_4h_5\left|\left[\begin{array}{cc}(\Pi^T\mathbf{l})_1 & (\Pi^T\mathbf{l})_2\\ (\Pi^T\mathbf{k})_1 & (\Pi^T\mathbf{k})_2\end{array}\right]\right|^2\end{aligned} \tag{6.21}$$

Now consider

$$\begin{aligned}\mathbf{l}^TA\mathbf{k} &= \mathbf{l}^T\Pi A_D\Pi^T\mathbf{k}\\ &= \alpha\left|\left[\begin{array}{cc}(\Pi^T\mathbf{l})_1 & (\Pi^T\mathbf{l})_2\\ (\Pi^T\mathbf{k})_1 & (\Pi^T\mathbf{k})_2\end{array}\right]\right|\end{aligned} \tag{6.22}$$

where A_D is the normal form of A

$$A_D=\left[\begin{array}{ccccc} 0 & \alpha & 0 & 0 & 0\\ -\alpha & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right] \tag{6.23}$$

corresponding to the diagonal form of h.

Combining these observations we obtain the attractive formula

$$(\mathbf{l}^TA\mathbf{k})^2=\lambda\left|\begin{array}{ccc} h & \mathbf{l} & \mathbf{k}\\ \mathbf{l^T} & 0 & 0\\ \mathbf{k^T} & 0 & 0\end{array}\right| \tag{6.24}$$

where λ is a function yet to be determined.

The undetermined factor can be found from a (simple) singularity argument and also by a more involved, algebraic expansion and as for genus two we will present the latter.

We return to the original relation on the curve for y_1 of degree 8 in x_1. By substituting for y_1 we obtain a sextic in x_1 from which we eliminate the degree 6 and 5 terms using the cubic expression (6.12). The resulting quartic identity in x_1 has leading term

$$\wp_{333}^2 + \frac{1}{4}\begin{vmatrix} h_{33} & h_{34} & h_{35} \\ h_{43} & h_{44} & h_{45} \\ h_{53} & h_{54} & h_{55} \end{vmatrix}. \tag{6.25}$$

Hence $\lambda = -\frac{1}{4}$ in the full quadratic identity:

$$(\mathbf{l}^T A\mathbf{k})^2 = -\frac{1}{4}\begin{vmatrix} h & \mathbf{l} & \mathbf{k} \\ \mathbf{l^T} & 0 & 0 \\ \mathbf{k^T} & 0 & 0 \end{vmatrix} \tag{6.26}$$

This formula is a new result of this paper.

7. Identities for $\wp_{ijkl}$

In all three cases discussed above there are identities for the four index $\wp$-functions of the form:

$$\wp_{ijkl} = F(\wp_{11}, \wp_{12}, \wp_{22}, \wp_{13}, \ldots) \tag{7.1}$$

that are obtained by differentiating the identities quadratic in the $\wp_{ijk}$ and (for genus two and three) using certain identities involving two and three index $\wp$-functions.

Clearly in genus one we get

$$\wp'' = 6\wp^2 - \frac{1}{2}(a_0a_4 - 4a_1a_3 + 3a_2^2) \tag{7.2}$$

recalling that the two index $\wp$ function is written as $\wp$ in this case.

In genus two we start with the identity for $\wp_{222}^2$. Differentiating,

$$\begin{aligned} -8\wp_{222}\wp_{2222} &= \begin{vmatrix} 4\wp_{112} & h_{23} & h_{24} \\ 2\wp_{122} & h_{33} & h_{34} \\ -2\wp_{222} & h_{43} & h_{44} \end{vmatrix} + \begin{vmatrix} h_{22} & 2\wp_{122} & h_{24} \\ h_{32} & 4\wp_{222} & h_{34} \\ h_{42} & 0 & h_{44} \end{vmatrix} \\ &\quad + \begin{vmatrix} h_{22} & h_{23} & -2\wp_{222} \\ h_{32} & h_{33} & 0 \\ h_{42} & h_{43} & 0 \end{vmatrix} \\ &= 4\wp_{112}\begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} + 4\wp_{122}\begin{vmatrix} h_{32} & h_{34} \\ h_{42} & h_{44} \end{vmatrix} \\ &\quad + 4\wp_{222}\left(-\begin{vmatrix} h_{23} & h_{24} \\ h_{33} & h_{34} \end{vmatrix} + \begin{vmatrix} h_{22} & h_{24} \\ h_{42} & h_{44} \end{vmatrix} \right) \end{aligned}$$

The first two terms on the right hand side can be replaced by a single term with factor $\wp_{222}$ by utilizing the identity

$$\wp_{112}\begin{vmatrix} h_{3,3} & h_{3,4} \\ h_{4,3} & h_{4,4} \end{vmatrix} + \wp_{122}\begin{vmatrix} h_{3,2} & h_{3,4} \\ h_{4,2} & h_{4,4} \end{vmatrix} + \wp_{222}\begin{vmatrix} h_{3,1} & h_{3,4} \\ h_{4,1} & h_{4,4} \end{vmatrix} = 0$$

This, in turn, is obtained from the quadratic identity by setting $l_i = h_{i+1,j}$ to get four identities of the form

$$h_{1,j}\wp_{222} - h_{2,j}\wp_{122} + h_{3,j}\wp_{112} - h_{4,j}\wp_{111} = 0 \tag{7.3}$$

and eliminating $\wp_{111}$ from the pair $j = 3, 4$. Thus

$$
\begin{aligned}
(7.4) \qquad -\wp_{2222} &= \frac{1}{2}\left(-\begin{vmatrix} h_{2,3} & h_{2,4} \\ h_{3,3} & h_{3,4} \end{vmatrix} + \begin{vmatrix} h_{2,2} & h_{2,4} \\ h_{4,2} & h_{4,4} \end{vmatrix} - \begin{vmatrix} h_{3,1} & h_{3,4} \\ h_{4,1} & h_{4,4} \end{vmatrix} \right) \\
\frac{1}{3}(-\wp_{2222} + 6\wp_{22}^2) &= a_2a_6 - 4a_3a_5 + 3a_4^2 + a_6\wp_{11} - 2a_5\wp_{12} + a_4\wp_{22}
\end{aligned}
$$

Application of $\mathbf{e}$ and $\mathbf{f}$ to this identity shows that it is highest weight for a five dimensional representation reproducing the classic partial differential equations of Baker [7].

The fully general, covariant genus three equations have not been written down before. We proceed as before by differentiating the $\wp_{333}^2$ relation with respect to u_3:

$$
\begin{aligned}
-8\wp_{333}\wp_{3333} &= \begin{vmatrix} 4\wp_{223} - 4\wp_{133} & h_{34} & h_{35} \\ 2\wp_{233} & h_{44} & h_{45} \\ -2\wp_{333} & h_{54} & h_{55} \end{vmatrix} + \begin{vmatrix} h_{33} & 2\wp_{233} & h_{35} \\ h_{43} & 4\wp_{333} & h_{45} \\ h_{53} & 0 & h_{55} \end{vmatrix} \\
&\quad + \begin{vmatrix} h_{33} & h_{34} & -2\wp_{333} \\ h_{43} & h_{44} & 0 \\ h_{53} & h_{54} & 0 \end{vmatrix} \\
&= 4(\wp_{223} - \wp_{133}) \begin{vmatrix} h_{44} & h_{45} \\ h_{54} & h_{55} \end{vmatrix} - 4\wp_{233} \begin{vmatrix} h_{34} & h_{35} \\ h_{54} & h_{55} \end{vmatrix} \\
&\quad + 4\wp_{333} \left(\begin{vmatrix} h_{33} & h_{35} \\ h_{53} & h_{55} \end{vmatrix} - \begin{vmatrix} h_{34} & h_{35} \\ h_{44} & h_{45} \end{vmatrix} \right)
\end{aligned}
$$

As before we can derive identities linear in the $\wp_{ijk}$ from the general quadratic identities. Putting $k_0 = 1$, $k_1 = k_2 = k_3 = k_4 = 0$ and $l_i = h_{i+1,j}$, $i = 0 \ldots 4$ gives, for any choice of j,

$$
(7.5) \qquad h_{2j}\wp_{333} - h_{3j}\wp_{233} + h_{4j}(\wp_{223} - \wp_{133}) - h_{5j}(\wp_{222} - 2\wp_{123}) = 0
$$

Eliminating the $\wp_{222} - 2\wp_{123}$ term between the cases $j = 4$ and $j = 5$ yields

$$
(7.6) \qquad (\wp_{223} - \wp_{133}) \begin{vmatrix} h_{44} & h_{45} \\ h_{54} & h_{55} \end{vmatrix} - \wp_{233} \begin{vmatrix} h_{34} & h_{35} \\ h_{54} & h_{55} \end{vmatrix} + \wp_{333} \begin{vmatrix} h_{24} & h_{25} \\ h_{54} & h_{55} \end{vmatrix} = 0
$$

Use of this identity in the equation for $\wp_{333}\wp_{3333}$ gives

$$
(7.7) \qquad -2\wp_{3333} = -\begin{vmatrix} h_{24} & h_{25} \\ h_{54} & h_{55} \end{vmatrix} + \begin{vmatrix} h_{33} & h_{35} \\ h_{53} & h_{55} \end{vmatrix} - \begin{vmatrix} h_{34} & h_{35} \\ h_{44} & h_{45} \end{vmatrix}
$$

and thus, by application of $\mathbf{f}$, the nine dimensional space of identities,

$$\begin{aligned}
-\wp_{3333}+6\wp_{33}^2 &= 10(a_4a_8-4a_5a_7+3a_6^2)\\
&\quad +8a_6\wp_{33}-8a_7wp_{23}+a_8(3\wp_{22}-4\wp_{13})\\
-\wp_{2333}+6\wp_{23}\wp_{33} &= 10(a_3a_8-3a_4a_7+2a_5a_6)\\
&\quad +12a_5\wp_{33}-10\wp_{23}+4a_7(\wp_{22}-3\wp_{13})\\
&\quad +2a_8\wp_{12}\\
2(-\wp_{1333}+6\wp_{13}\wp_{33}) &= 10(3a_2a_8-4a_3a_7-11a_4a_6+12a_5^2)\\
+3(-\wp_{2233}+2\wp_{22}\wp_{33}+4\wp_{23}^2) &\quad +60a_4\wp_{33}-36a_5\wp_{23}-2a_6(9\wp_{22}-52\wp_{13})\\
&\quad +20a_7\wp_{12}+4a_8\wp_{11}\\
-\wp_{2223}+6\wp_{22}\wp_{23} &= 10(a_1a_8+2a_2a_7-12a_3a_6+9a_4a_5)\\
+3(-\wp_{1233}+2\wp_{12}\wp_{33}+4\wp_{13}\wp_{23}) &\quad +40a_3\wp_{33}-10a_4\wp_{23}+4a_5(3\wp_{22}-29\wp_{13})\\
&\quad +18a_6\wp_{12}+12a_7\wp_{11}\\
-\wp_{2222}+6\wp_{22}^2 &= 10(a_0a_8+12a_1a_7-22a_2a_6-36a_3a_5+45a_4^2\\
+6(-\wp_{1133}+2\wp_{11}\wp_{33}+4\wp_{13}^2) &\quad +120a_2\wp_{33}+40a_3\wp_{23}+50a_4(\wp_{22}-12\wp_{13})\\
+12(-\wp_{1223}+4\wp_{12}\wp_{23}+2\wp_{13}\wp_{22}) &\quad +40a_5\wp_{12}+120a_6\wp_{11}\\
-\wp_{1222}+6\wp_{12}\wp_{22} &= 10(a_0a_7+2a_1a_6-12a_2a_5+9a_3a_4)\\
+3(-\wp_{1123}+4\wp_{12}\wp_{13}+2\wp_{11}\wp_{23}) &\quad +12a_1\wp_{33}+18a_2\wp_{23}+4a_3(3\wp_{22}-29\wp_{13})\\
&\quad -10a_4\wp_{12}+40a_5\wp_{11}\\
2(-\wp_{1113}+6\wp_{11}\wp_{13}) &= 10(3a_0a_6-4a_1a_5-11a_2a_4+12a_3^2)\\
+3(-\wp_{1122}+2\wp_{11}\wp_{22}+4\wp_{12}^2) &\quad +4a_0\wp_{33}+20a_1\wp_{23}+2a_2(9\wp_{22}-52\wp_{13})\\
&\quad -36a_3\wp_{12}+60a_4\wp_{11}\\
-\wp_{1112}+6\wp_{11}\wp_{12} &= 10(a_0a_5-3a_1a_4+2a_2a_3)\\
&\quad +2a_0\wp_{23}+4a_1(\wp_{22}-3\wp_{13})-10a_2\wp_{12}\\
&\quad +12a_3\wp_{11}\\
-\wp_{1111}+6\wp_{11}^2 &= 10(a_0a_4-4a_1a_3+3a_2^2)\\
&\quad +a_0(3\wp_{22}-4\wp_{13})-8a_1\wp_{12}+8a_2\wp_{11}
\end{aligned}$$

Given that there are fifteen of the symbols $\wp_{ijkl}$ we expect to be able to find a further six identities.

Thus, returning to the $\wp_{333}^2$ identity and differentiating with respect to u_1 this time yields

$$\begin{aligned}
-8\wp_{333}\wp_{1333} &= \begin{vmatrix} 4\wp_{122}-4\wp_{113} & h_{34} & h_{35}\\ 2\wp_{123} & h_{44} & h_{45}\\ -2\wp_{133} & h_{54} & h_{55}\end{vmatrix} + \begin{vmatrix} h_{33} & 2\wp_{123} & h_{35}\\ h_{43} & 4\wp_{133} & h_{45}\\ h_{53} & 0 & h_{55}\end{vmatrix}\\
&\quad + \begin{vmatrix} h_{33} & h_{34} & -2\wp_{333}\\ h_{43} & h_{44} & 0\\ h_{53} & h_{54} & 0\end{vmatrix}\\
&= 4(\wp_{122}-\wp_{113})\begin{vmatrix} h_{44} & h_{45}\\ h_{54} & h_{55}\end{vmatrix} - 4\wp_{123}\begin{vmatrix} h_{34} & h_{35}\\ h_{54} & h_{55}\end{vmatrix}\\
&\quad +4\wp_{133}\left(\begin{vmatrix} h_{33} & h_{35}\\ h_{53} & h_{55}\end{vmatrix} - \begin{vmatrix} h_{34} & h_{35}\\ h_{44} & h_{45}\end{vmatrix}\right)
\end{aligned}$$

This time some appropriate identities arise by choosing $k_0 = 0, k_1 = 1, k_2 = 0, k_3 = 0$ and $k_4 = 0$ and the l_i as before:

$$h_{ij}\wp_{333} - h_{3j}\wp_{133} + h_{4j}\wp_{123} - h_{5j}(\wp_{122} - \wp_{113}) = 0 \tag{7.8}$$

These allow us to replace the terms on the right hand side of the $\wp_{1333}$ equation by terms involving $\wp_{333}$ and so factor this out to leave

$$-2\wp_{1333} = -\begin{vmatrix} h_{14} & h_{15} \\ h_{44} & h_{45} \end{vmatrix} + \begin{vmatrix} h_{13} & h_{53} \\ h_{15} & h_{55} \end{vmatrix} \tag{7.9}$$

Applying $\mathbf{e}$ to this identity gives the $\wp_{2333}$ identity found above. Successive applications of $\mathbf{f}$ however yield a set of seven identities:

$$\begin{aligned}
-\wp_{1333} + 6\wp_{13}\wp_{33} &= 3a_2a_8 - 8a_3a_7 + 5a_4a_6 \\
&\quad +3a_4\wp_{33} - 10a_6\wp_{13} + 4a_7\wp_{12} - a_8\wp_{11} \\
-\wp_{1233} + 2\wp_{12}\wp_{33} + 4\wp_{13}\wp_{23} &= 2a_1a_8 - 12a_3a_6 + 10a_4a_5 \\
&\quad +4a_3\wp_{33} + 2a_4\wp_{23} - 20a_5\wp_{13} + 6a_6\wp_{12} \\
-\wp_{1133} + 2\wp_{11}\wp_{33} + 4\wp_{13}^2 &= a_0a_8 + 8a_1a_7 - 18a_2a_6 - 16a_3a_5 + 25a_4^2 \\
-\wp_{1223} + 2\wp_{13}\wp_{22} + 4\wp_{12}\wp_{23} &\quad +6a_2\wp_{33} + 8a_3\wp 23 + a_4(\wp_{22} - 48\wp_{13}) \\
&\quad +8a_5\wp_{12} + 6a_6\wp_{11} \\
-\wp_{1222} + 6\wp_{12}\wp_{22} &= 16a_0a_7 + 20a_1a_6 - 156a_2a_5 + 120a_3a_4 \\
+6(-\wp_{1123} + 2\wp_{11}\wp_{23} + 4\wp_{12}\wp_{13}) &\quad +12a_1\wp_{33} + 36a_2\wp_{23} + 4(3a_3\wp_{22} - 44\wp_{13}) \\
&\quad -4a_4\wp_{12} + 52a_5\wp_{11} \\
-\wp_{1113} + 6\wp_{11}\wp_{12} &= 11a_0a_6 - 16a_1a_5 - 35a_2a_4 + 40a_3^2 \\
-\wp_{1122} + 2\wp_{11}\wp_{22} + 4\wp_{12}^2 &\quad +a_0\wp_{33} + 8a_1\wp_{23} + 2a_2(3\wp_{22} - 19\wp_{13}) \\
&\quad -12a_3\wp_{12} + 21a_4\wp_{11} \\
-\wp_{1112} + 6\wp_{11}\wp_{12} &= 10(a_0a_5 - 3a_1a_4 + 2a_2a_3) \\
&\quad +2a_0\wp_{23} + 4a_1(\wp_{22} - 3\wp_{13}) - 10a_2\wp_{12} \\
&\quad +12a_3\wp_{11} \\
-\wp_{1111} + 6\wp_{11}^2 &= 10(a_0a_4 - 4a_1a_3 + 3a_2^2) \\
&\quad +a_0(3\wp_{22} - 4\wp_{13}) - 8a_1\wp_{12} + 8a_2\wp_{11}
\end{aligned}$$

Of these seven the last two are already represented in the previous set so that we still seek another one. To find this go to the quadratic identity for $\wp_{133}$,

$$-\wp_{133}^2 = \frac{1}{4}\begin{vmatrix} h_{11} & h_{14} & h_{15} \\ h_{41} & h_{44} & h_{55} \\ h_{51} & h_{54} & h_{55} \end{vmatrix} \tag{7.10}$$

and differentiate with respect to u_1. Using similar identities to before we find

$$-2\wp_{1133} = \begin{vmatrix} h_{1,1} & h_{1,5} \\ h_{5,1} & h_{5,5} \end{vmatrix} - \begin{vmatrix} h_{1,4} & h_{1,5} \\ h_{2,4} & h_{2,5} \end{vmatrix} \tag{7.11}$$

and applying $\mathbf{f}$ successively arrive at:

$$
\begin{aligned}
2(-\wp_{1133}+6\wp_{13}^2)+4(\wp_{23}\wp_{12}-\wp_{13}\wp_{22}) &= a_0a_8-16a_3a_5+15a_4^2 \\
&\quad +8a_3\wp_{23}-2a_4(\wp_{22}+12\wp_{13})+8a_5\wp_{12} \\
-\wp_{1123}+4\wp_{12}\wp_{13}+2\wp_{23}\wp_{11} &= 2a_0a_7-12a_2a_5+10a_3a_4 \\
&\quad +6a_2\wp_{23}-20a_3\wp_{13}+2a_4\wp_{12}+4a_5\wp_{11} \\
-\wp_{1122}+2\wp_{11}\wp_{22}+4\wp_{12}^2 &= 14a_0a_6-24a_1a_5-30a_2a_4+40a_3^3 \\
+2(-\wp_{1113}+6\wp_{11}\wp_{13}) &\quad +12a_1\wp_{23}+6a_2(\wp_{22}-8\wp_{13}) \\
&\quad -12a_3\wp_{12}+24a_4\wp_{11} \\
-\wp_{1112}+6\wp_{11}\wp_{12} &= 10a_0a_5-30a_1a_4+20a_2a_3 \\
&\quad +2a_0\wp_{23}+4a_1(\wp_{22}-3\wp_{13}) \\
&\quad -10a_2\wp_{12}+12a_3\wp_{11} \\
-\wp_{1111}+6\wp_{11}^2 &= 10a_0a_4-40a_1a_3+30a_2^2 \\
&\quad +a_0(3\wp_{22}-4\wp_{13})-8a_1\wp_{12}+8a_2\wp_{11}
\end{aligned}
$$

Only one of these is linearly independent of the identities we already have.

In Appendix 1 we summarize these identities and in Appendix 2 we compare them with the original, non-covariant identities of Baker [6], showing that they are equivalent under a simple transformation. To this end the identities in Appendix 1 are written in a Baker friendly form where each involves but one of the four index objects. This is not ideal from the representation theoretic viewpoint however, as the identities then do not fall naturally into multiplets.

8. Conclusions

This paper establishes that the use of covariant methods for hyperelliptic curves is a practical tool in the construction and understanding of the partial differential equations satisfied by the $\wp$-function. In order to do so the definition of the $\wp$ function has to be slightly modified in a way that does not alter its analytic properties. The resulting covariant identities for the $\wp_{ijk}$ and $\wp_{ijkl}$ (Appendix 1) differ in detail from those obtained by Baker (Appendix 2) but are generic and are derived in a straight forward, economical way with minimal use of computer algebra and in an algorithmic manner. Because the equivalence of the two sets of equations is by no means self evident we also specify in Appendix 2 the transformation between the two definitions of the $\wp_{ij}$. Given Baker's equations one could have written down the covariant genus three equations by deducing this simple transformation by comparing the classical and covariant polar forms. But this would not have been a test of the machinery nor would it have provided us with the neat expression for the quadratic, genus three identities.

By "minimal use of computer algebra" we mean that the derivation of the highest weight identities was carried out by hand. A computer algebra programme was used to implement the actions of $\mathbf{e}$ and $\mathbf{f}$ on these highest weight identities in order to check covariance and to generate the full sets of identities. The other use of computer algebra, as remarked at the time, was in rearranging by linear superposition, the identities linear in the $\wp_{ijk}$ in the genus three case, into the form (6.18).

It may be remarked that Baker's equations are a little simpler than the covariant ones. From the current point of view this is a simplification bought at the expense of

the more abstract simplification which incorporates the representation theory. The drawback of the simplicity is that each identity has to be obtained independently. The advantage of the marginally more involved covariant set is that the fifteen identities for the $\wp_{ijkl}$ decompose into sets of nine, five and one elements from each of which one need only find a single identity using the singularity analysis, the others following by application of the *raising* and *lowering operators*, $\mathbf{e}$ and $\mathbf{f}$.

Further, the representation theory lays bare a pattern of bones with further intriguing symmetries that beg further study, particularly in view of the σ function and hyperelliptic addition laws. However, the most pressing issue now is to apply these methods to the more difficult non-hyperelliptic curves of low genus.

9. Appendix 1

Here we summarize the four-index relations for the covariant $\wp$-function in the genus three case. Δ denotes a quadratic in two index functions: $\Delta = \wp_{11}\wp_{33} -$

$\wp_{12}\wp_{23} - \wp_{13}^2 + \wp_{13}\wp_{22}$.

$$
\begin{aligned}
-\wp_{3333} + 6\wp_{33}^2 &= 8a_6\wp_{33} - 8a_7\wp_{23} + a_8(3\wp_{22} - 4\wp_{13}) \\
&\quad +10(a_4a_8 - 4a_5a_7 + 3a_6^2) \\
-\wp_{2333} + 6\wp_{23}\wp_{33} &= 12a_5\wp_{33} - 10a_6\wp_{23} + 4a_7(\wp_{22} - 3\wp_{13}) \\
&\quad +2a_8\wp_{12} \\
&\quad +10(a_3a_8 - 3a_4a_7 + 2a_5a_6) \\
-\wp_{2233} + 4\wp_{23}^2 + 2\wp_{22}\wp_{33} &= 18a_4\wp_{33} - 12a_5\wp_{23} + 2a_6(3\wp_{22} - 14\wp_{13}) \\
&\quad +4a_7\wp_{12} + 2a_8\wp_{11} \\
&\quad +8(a_2a_8 - a_3a_7 - 5a_4a_6 + 5a_5^2) \\
-\wp_{2223} + 6\wp_{22}\wp_{23} &= 28a_3\wp_{33} - 16a_4\wp_{23} + 4a_5(3\wp_{22} - 14\wp_{13}) \\
&\quad +12a_7\wp_{11} \\
&\quad +4(a_1a_8 + 5a_2a_7 - 21a_3a_6 + 15a_4a_5) \\
-\wp_{2222} + 6\wp_{22}^2 - 12\Delta &= 48a_3\wp_{33} - 32a_3\wp_{23} + 32a_4(\wp_{22} - 3\wp_{13}) \\
&\quad -32a_5\wp_{12} + 48a_6\wp_{11} \\
&\quad a_0a_8 + 24a_1a_7 - 4a_2a_6 - 216a_3a_5 + 195a_4^2 \\
-\wp_{1333} + 6\wp_{13}\wp_{33} &= 3a_4\wp_{33} - 10a_6\wp_{13} + 4a_7\wp_{12} - a_8\wp_{11} \\
&\quad +3a_2a_8 - 8a_3a_7 + 5a_4a_6 \\
-\wp_{1233} + 4\wp_{13}\wp_{23} + 2\wp_{12}\wp_{33} &= 4a_3\wp_{33} + 2a_4\wp_{23} - 20a_5\wp_{13} + 6a_6\wp_{12} \\
&\quad +2(a_1a_8 - 6a_3a_6 + 5a_4a_5) \\
-\wp_{1223} + 4\wp_{12}\wp_{23} + 2\wp_{13}\wp_{22} &= 6a_2\wp_{33} + 4a_3\wp_{23} + 2a_4(\wp_{22} - 18\wp_{13}) \\
+2\Delta &\quad +4a_5\wp_{12} + 6a_6\wp_{11} \\
&\quad +\frac{1}{2}(a_0a_8 + 16a_1a_7 - 36a_2a_6 - 16a_3a_5 + 35a_4^2) \\
-\wp_{1222} + 6\wp_{12}\wp_{22} &= 12a_1\wp_{33} + 4a_3(3\wp_{22} - 14\wp_{13}) \\
&\quad -16a_4\wp_{12} + 28a_5\wp_{11} \\
&\quad +4(a_0a_7 + 5a_1a_6 - 21a_2a_5 + 15a_3a_4) \\
-\wp_{1133} + 4\wp_{13}^2 + 2\wp_{11}\wp_{33} - 2\Delta &= 4a_3\wp_{23} - a_4(\wp_{22} - 12\wp_{13}) + 4a_5\wp_{12} \\
&\quad +\frac{1}{2}(a_0a_8 - 16a_3a_5 + 15a_4^2) \\
-\wp_{1123} + 4\wp_{12}\wp_{13} + 2\wp_{11}\wp_{23} &= 6a_2\wp_{23} - 20a_3\wp_{13} + 2a_4\wp_{12} + 4a_5\wp_{11} \\
&\quad +2(a_0a_7 - 6a_2a_5 + 5a_3a_4) \\
-\wp_{1122} + 4\wp_{12}^2 + 2\wp_{11}\wp_{22} &= 2a_0\wp_{33} + 4a_1\wp_{23} + 2a_2(3\wp_{22} - 14\wp_{13}) \\
&\quad -12a_3\wp_{12} + 18a_4\wp_{11} \\
&\quad +8(a_0a_6 - a_1a_5 - 5a_2a_4 + 5a_3^2) \\
-\wp_{1113} + 6\wp_{11}\wp_{13} &= -a_0\wp_{33} + 4a_1\wp_{23} - 10a_2\wp_{13} + 3a_4\wp_{11} \\
&\quad +3a_0a_6 - 8a_1a_5 + 5a_2a_4 \\
-\wp_{1112} + 6\wp_{11}\wp_{12} &= 2a_0\wp_{23} + 4a_1(\wp_{22} - 3\wp_{13}) \\
&\quad -10a_2\wp_{12} + 12a_3\wp_{11} \\
&\quad +10(a_0a_5 - 3a_1a_4 + 2a_2a_3) \\
-\wp_{1111} + 6\wp_{11}^2 &= a_0(3\wp_{22} - 4\wp_{13}) - 8a_1\wp_{12} + 8a_2\wp_{11} \\
&\quad +10(a_0a_4 - 4a_1a_3 + 3a_2^2)
\end{aligned}
$$

10. Appendix 2

Here we reproduce the genus three equations from Baker's paper [6]. We will denote by $\wp^{\mathfrak{B}}$ the traditional genus three $\wp$-function. For ease of comparison we have also rewritten the λ_i coefficients of the monomials in x in the octic curve in Baker's paper in terms of the a_i used above.

$$
\begin{aligned}
{\wp^{\mathfrak{B}}}_{3333} - 6{\wp^{\mathfrak{B}}}^2_{33} &= 28a_6{\wp^{\mathfrak{B}}}_{33} + 8a_7{\wp^{\mathfrak{B}}}_{23} \\
&\quad +a_8(4{\wp^{\mathfrak{B}}}_{13} - 3{\wp^{\mathfrak{B}}}_{22}) \\
&\quad -35a_4a_8 + 56a_5a_7 \\
{\wp^{\mathfrak{B}}}_{2333} - 6{\wp^{\mathfrak{B}}}_{23}{\wp^{\mathfrak{B}}}_{33} &= 28a_6{\wp^{\mathfrak{B}}}_{23} + 4a_7(3{\wp^{\mathfrak{B}}}_{13} - {\wp^{\mathfrak{B}}}_{22}) \\
&\quad +2a_8{\wp^{\mathfrak{B}}}_{12} - 14a_3a_8 \\
{\wp^{\mathfrak{B}}}_{2233} - 4{\wp^{\mathfrak{B}}}^2_{23} - 2{\wp^{\mathfrak{B}}}_{22}{\wp^{\mathfrak{B}}}_{33} &= 28a_5{\wp^{\mathfrak{B}}}_{23} + 28a_6{\wp^{\mathfrak{B}}}_{13} - 4a_7{\wp^{\mathfrak{B}}}_{12} \\
&\quad -2a_8{\wp^{\mathfrak{B}}}_{11} - 14a_2a_8 \\
{\wp^{\mathfrak{B}}}_{2223} - 6{\wp^{\mathfrak{B}}}_{22}{\wp^{\mathfrak{B}}}_{23} &= -28a_3{\wp^{\mathfrak{B}}}_{33} + 7 - a_4{\wp^{\mathfrak{B}}}_{23} + 56a_5{\wp^{\mathfrak{B}}}_{13} \\
&\quad -12a_7{\wp^{\mathfrak{B}}}_{11} \\
&\quad -4a_1a_8 - 56a_2a_7 \\
{\wp^{\mathfrak{B}}}_{2222} - 6{\wp^{\mathfrak{B}}}^2_{22} - 12\Delta &= -84a_2{\wp^{\mathfrak{B}}}_{33} + 56a_3{\wp^{\mathfrak{B}}}_{23} + 70a_4{\wp^{\mathfrak{B}}}_{22} \\
&\quad +56a_5{\wp^{\mathfrak{B}}}_{12} - 84a_6{\wp^{\mathfrak{B}}}_{11} \\
&\quad -392a_2a_6 + 392a_3a_5 \\
{\wp^{\mathfrak{B}}}_{1333} - 6{\wp^{\mathfrak{B}}}_{13}{\wp^{\mathfrak{B}}}_{33} &= 28a_6{\wp^{\mathfrak{B}}}_{13} - 4a_7{\wp^{\mathfrak{B}}}_{14} + a_8{\wp^{\mathfrak{B}}}_{11} \\
{\wp^{\mathfrak{B}}}_{1233} - 4{\wp^{\mathfrak{B}}}_{13}{\wp^{\mathfrak{B}}}_{23} - 2{\wp^{\mathfrak{B}}}_{12}{\wp^{\mathfrak{B}}}_{33} &= 28a_5{\wp^{\mathfrak{B}}}_{13} - 2a_1a_8 \\
{\wp^{\mathfrak{B}}}_{1223} - 4{\wp^{\mathfrak{B}}}_{12}{\wp^{\mathfrak{B}}}_{23} - 2{\wp^{\mathfrak{B}}}_{13}{\wp^{\mathfrak{B}}}_{22} &= 70a_4{\wp^{\mathfrak{B}}}_{13} - 8a_1a_7 - \frac{1}{2}a_0a_8 \\
+2\Delta & \\
{\wp^{\mathfrak{B}}}_{1222} - 6{\wp^{\mathfrak{B}}}_{12}{\wp^{\mathfrak{B}}}_{22} &= -12a_1{\wp^{\mathfrak{B}}}_{33} + 56a_3{\wp^{\mathfrak{B}}}_{13} + 70a_4{\wp^{\mathfrak{B}}}_{12} \\
&\quad -28a_5{\wp^{\mathfrak{B}}}_{11} \\
&\quad -112a_1a_6 - 4a_0a_7 \\
{\wp^{\mathfrak{B}}}_{1133} - 4{\wp^{\mathfrak{B}}}^2_{13} - 2{\wp^{\mathfrak{B}}}_{11}{\wp^{\mathfrak{B}}}_{33} &= -\frac{1}{2}a_1a_8 \\
-2\Delta & \\
{\wp^{\mathfrak{B}}}_{1123} - 4{\wp^{\mathfrak{B}}}_{12}{\wp^{\mathfrak{B}}}_{13} - 2{\wp^{\mathfrak{B}}}_{11}{\wp^{\mathfrak{B}}}_{23} &= 28a_3{\wp^{\mathfrak{B}}}_{13} - 2a_0a_7 \\
{\wp^{\mathfrak{B}}}_{1122} - 4{\wp^{\mathfrak{B}}}^2_{12} - 2{\wp^{\mathfrak{B}}}_{11}{\wp^{\mathfrak{B}}}_{22} &= -2a_0{\wp^{\mathfrak{B}}}_{33} - 4a_1{\wp^{\mathfrak{B}}}_{23} + 28a_2{\wp^{\mathfrak{B}}}_{13} \\
&\quad +28a_3{\wp^{\mathfrak{B}}}_{12} - 14a_0a_6 \\
{\wp^{\mathfrak{B}}}_{1113} - 6{\wp^{\mathfrak{B}}}_{11}{\wp^{\mathfrak{B}}}_{13} &= a_0{\wp^{\mathfrak{B}}}_{33} - 4a_1{\wp^{\mathfrak{B}}}_{23} + 28a_2{\wp^{\mathfrak{B}}}_{13} \\
{\wp^{\mathfrak{B}}}_{1112} - 6{\wp^{\mathfrak{B}}}_{11}{\wp^{\mathfrak{B}}}_{12} &= -2a_0{\wp^{\mathfrak{B}}}_{23} + 4a_1(3{\wp^{\mathfrak{B}}}_{13} - {\wp^{\mathfrak{B}}}_{22}) \\
&\quad +28a_2{\wp^{\mathfrak{B}}}_{12} - 14a_0a_5 \\
{\wp^{\mathfrak{B}}}_{1111} - 6{\wp^{\mathfrak{B}}}^2_{11} &= a_0(4\wp_{13} - 3{\wp^{\mathfrak{B}}}_{22}) + 8a_1{\wp^{\mathfrak{B}}}_{12} \\
&\quad +28a_2{\wp^{\mathfrak{B}}}_{11} \\
&\quad -35a_0a_4 + 56a_1a_3
\end{aligned}
$$

The Baker equivalent of the 5×5 matrix h we will call $h^{\mathfrak{B}}$ and since our covariant form is to be replaced by the classical polar form $h^{\mathfrak{B}}$ will be given by:

$$\left[\begin{array}{ccccc} a_0 & 4a_1 & -2\wp^{\mathfrak{B}}{}_{11} & -2\wp^{\mathfrak{B}}{}_{12} & -2\wp^{\mathfrak{B}}{}_{13} \\ 4a_1 & 28a_2+4\wp^{\mathfrak{B}}{}_{11} & 28a_3+2\wp^{\mathfrak{B}}{}_{12} & -2\wp^{\mathfrak{B}}{}_{22}+4\wp^{\mathfrak{B}}{}_{13} & -2\wp^{\mathfrak{B}}{}_{23} \\ -2\wp^{\mathfrak{B}}{}_{11} & 28a_3+2\wp^{\mathfrak{B}}{}_{12} & 70a_4+4\wp^{\mathfrak{B}}{}_{22}-4\wp^{\mathfrak{B}}{}_{13} & 28a_5+2\wp^{\mathfrak{B}}{}_{23} & -2\wp^{\mathfrak{B}}{}_{33} \\ -2\wp^{\mathfrak{B}}{}_{12} & -2\wp^{\mathfrak{B}}{}_{22}+4\wp^{\mathfrak{B}}{}_{13} & 28a_5+2\wp^{\mathfrak{B}}{}_{23} & 28a_6+4\wp^{\mathfrak{B}}{}_{33} & 4a_7 \\ -2\wp^{\mathfrak{B}}{}_{13} & -2\wp^{\mathfrak{B}}{}_{23} & -2\mathfrak{B}_{33} & 4a_7 & a_8 \end{array}\right] \tag{10.1}$$

Consequently

$$\begin{aligned} \wp^{\mathfrak{B}}{}_{11} &= \wp_{11}-3a_2 \\ \wp^{\mathfrak{B}}{}_{12} &= \wp_{12}-2a_3 \\ \wp^{\mathfrak{B}}{}_{13} &= \wp_{13}-\frac{1}{2}a_4 \\ \wp^{\mathfrak{B}}{}_{22} &= \wp_{22}-9a_4 \\ \wp^{\mathfrak{B}}{}_{23} &= \wp_{23}-2a_5 \\ \wp^{\mathfrak{B}}{}_{33} &= \wp_{33}-3a_6 \end{aligned}$$

(10.2)

Substitution for either $\wp$ or $\wp^{\mathfrak{B}}$ does indeed transform the two sets of equations into one another.

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